Existence and Uniqueness for a Crystalline Mean Curvature Flow

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Abstract

An existence and uniqueness result, up to fattening, for a class of crystalline mean curvature flows with natural mobility is proved. The results are valid in any dimension and for arbitrary, possibly unbounded, initial closed sets. The comparison principle is obtained by means of a suitable weak formulation of the flow, while the existence of a global-in-time solution follows via a minimizing movements approach.

1 Introduction

In this paper we consider the anisotropic mean curvature motion, that is, a flow of sets $t \mapsto E(t)$ (formally) governed by the law

$$V(x,t) = -m(\nu^{E(t)})\kappa_\phi^{E(t)}(x), \tag{1}$$

where $V(x,t)$ stands for the (outer) normal velocity of the boundary $\partial E(t)$ at x, ϕ is a given norm on $\mathbb{R}^N$ representing the *surface tension*, $\kappa_\phi^{E(t)}$ is the *anisotropic mean curvature* of $\partial E(t)$ associated with the anisotropy ϕ, and m is a positive *mobility* which depends on the outer unit normal $\nu^{E(t)}$ to $\partial E(t)$. Such an evolution law may be regarded as the gradient flow (with respect to a suitable formal Riemannian structure) of the anisotropic perimeter functional

$$P_\phi(E) = \int_{\partial E} \phi(\nu^E)\, d\mathcal{H}^{N-1}, \tag{2}$$

the anisotropic curvature κ_ϕ^E of ∂E being nothing but the first variation of (2) at E. When ϕ is differentiable in $\mathbb{R}^N \setminus \{0\}$, then κ_ϕ^E is given by

$$\kappa_\phi^E = \mathrm{div}\,\left(\nabla\phi(\nu^E)\right)\,. \tag{3}$$

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However, we are particularly interested in the case when ϕ is not differentiable, for instance the *crystalline* case, when the unit ball $B_\phi := \{\phi \leq 1\}$, known as the *Frank diagram*, is a convex polytope. In the latter case, we will only consider the natural mobility given by $m = \phi$. With this choice, (1) has the interesting property that the flow starting from a *Wulff shape*, that is, a level set of the polar ϕ° of ϕ, consists in a one-parameter family of shrinking Wulff shapes that extinguish in finite time. We recall that Wulff shapes are the only solutions to the isoperimetric problem associated with P_ϕ (see [29]).

The law (1) is used to describe several phenomena in Materials Science and Crystal Growth, see for instance [48, 38]. From the mathematical point of view, the geometric motion is well defined in a classical sense in the smooth case, that is, when ϕ is at least $C^{3,\alpha}$ (as well as the initial surface, and except at the origin) and "elliptic" (which means for instance that ϕ^2 is strongly convex) [2]. Of course, the classical mean curvature flow falls within this class and corresponds to the choice ϕ = Euclidean norm. In the smooth case, the main mathematical difficulties are related to the fact that singularities (like pinching) may form in finite time (see for instance [37]) in dimensions $N \geq 3$. Thus, the strong formulation of (1), which requires smoothness of the evolving sets, is well defined only for short times and one needs a weaker notion of solution that can handle the presence of singularities in order to define the flow for all positive times. When ϕ is smooth, this task has been already accomplished and different approaches have been proposed in the literature, starting from the pioneering work by Brakke [14], who suggested a weak formulation of the motion by mean curvature yielding deep regularity results but lacking uniqueness. These uniqueness issues have been subsequently overcome via the so-called *level set approach* [45, 27, 21, 34]. In particular, the case of (1) for m, ϕ of class C^2 is covered by [21]. The main idea is to represent the initial set as the zero sublevel set of a function u_0 and then to let all these level sets evolve according to the same geometric law (which makes sense thanks to the fact that the evolutions which we consider preserve inclusion). This procedure defines a time-dependent function $u(x,t)$ and transforms the geometric equation into a (degenerate) parabolic equation for u, which is shown to admit a *unique viscosity solution* with the prescribed initial datum u_0. The evolution of the zero sublevel set of such a solution defines a *generalized motion* (see also [10]), which exists for all times and agrees with the classical one for short times, before the appearance of singularities (see [28]). Such a motion satisfies a comparison principle and is unique whenever the level sets of u have zero Lebesgue measure, i.e., whenever the so-called fattening phenomenon does not occurs. Fattening may in fact appear even for a smooth initial datum E^0 (see [7]), but its occurrence is in some sense very "rare": for instance, it is easy to understand that almost all the sublevels sets of the signed distance function from any given set E^0 will not generate any fattening.

A third approach is represented by the minimizing movements scheme devised by Almgren, Taylor and Wang [2] and Luckhaus and Sturzenhecker [42]. It consists in constructing a sequence of discrete-in-time evolutions by iteratively solving suitable incremental minimum problems. Any limit of these evolutions as the time step vanishes defines a motion, which exists for all positive times (and is shown to be Hölder-continuous in time for the L^1 norm) and is usually referred to as a "flat flow". The connections between the generalized level set motion and Brakke solutions has been investigated in [39]. A simple proof of convergence of the Almgren-Taylor-Wang (ATW) to the generalized motion is shown in [20], while a consistency result was already shown in [2]. See also [24] for a similar convergence proof in a more general setting (allowing for unbounded surfaces, as in the present paper), and [41] for new proofs and a generalization to partitions of a related, but different minimizing movements scheme. Roughly speaking, it turns out that whenever fattening does not occur, the generalized level set motion coincides with the ATW flat flow and is also a solution in the sense of Brakke.

Let us now consider a crystalline anisotropy. This case is more difficult, due to the lack of smoothness in the involved differential operators. Indeed, the crystalline normal $\nabla\phi(\nu^E)$ is not uniquely defined for some directions and one needs to look at suitable selections of the (multivalued) subdifferential map, that is, vector fields $z : \partial E \to \mathbb{R}^N$, such that $z(x) \in \partial\phi(\nu^E(x))$ for a.e. x. If there exists an admissible field z with tangential divergence $\operatorname{div}_\tau z$ in $L^2(\partial E)$, then the crystalline curvature is given by the tangential divergence of z, where $\operatorname{div}_\tau z$ has minimal L^2-norm among all admissible fields (see [12, 32]). In particular, the crystalline curvature has a nonlocal character.

Showing (even local-in-time) existence and uniqueness for crystalline mean curvature flows is somewhat harder and still largely open. In dimension 2, the problem has been settled by developing a crystalline version of the viscosity approach for the level-set equation, see [31]. If the initial set is itself an appropriate planar crystal, the evolution equation boils down to a system of ODEs which has been studied in many former works, see in particular [3, 8, 30, 35], while existence and uniqueness of strong solutions for initial "regular" (in an appropriate sense) sets was shown recently in [17]. One advantage of the level-set approach of [31] is the ability to address much more general equations where the speed depends on the crystalline curvature and the normal in a nonlinear way. This construction has been extended to the dimension 3 in a recent work of Giga and Pozar [33], which was released while this paper was under review.

In dimension $N \geq 3$, apart for the new result just mentioned, the only general available notion of global-in-time solution we are aware of is the minimizing movements motion provided by the ATW scheme; however, no general comparison results have been established so far. In fact, higher-dimensional uniqueness results deal with special classes of initial data (for instance convex initial data as in [15, 13] or polyhedral sets as in [36]) or with very specific anisotropies (see [32] where a comparison principle valid in all dimensions has been established for the anisotropy $\phi(\nu) = |\nu'| + |\nu_N|$, with $\nu_N := \nu \cdot e_N$ and $|\nu'|$ the Euclidean norm of the orthogonal projection of ν onto $e_N^\perp$).

In this paper we prove a global-in-time existence and uniqueness (up to possible fattening) result for the crystalline mean curvature flow *valid in all dimensions, for arbitrary (possibly unbounded) initial sets, and for general crystalline anisotropies* ϕ, but under the particular choice $m = \phi$ in (1). We do so by providing a suitable weak formulation of the problem and then by showing that such a notion yields a comparison principle. We then implement a variant of the ATW scheme to establish an existence result.

Le us describe our approach in more details. It is based on ideas of [47, 10, 6, 15] (see also [27]). In order to motivate our formulation, let us assume for a moment that ϕ is smooth and that $t \mapsto E(t)$ is a regular flow obeying (1). Set $d(\cdot, t) := \mathrm{dist}(\cdot, E(t))$, where dist denotes the distance induced by the polar norm ϕ° (see (6) and (8) below). Then it is easy to see that the time partial derivative $\partial_t d$ of d on $\partial E(t)$ equals $-V/\phi(\nu^{E(t)})$, with V denoting the outer normal velocity of the moving boundary. The quantity $V/\phi(\nu^{E(t)})$ is nothing but the speed of the moving boundary along the *Cahn-Hoffmann* normal $\nabla\phi(\nu^{E(t)})$, see [38, 12]. By the above observations, (1) may be rewritten as

$$\partial_t d = \kappa_\phi^{E(t)} = \mathrm{div}\,(\nabla\phi(\nabla d)) \qquad \text{on } \partial E(t) = \partial\{d(\cdot, t) = 0\}.$$

(Here and throughout the paper ∇ stands for the spatial gradient.) On the other hand, if we look at a positive s-level set of d, the (weighted) normal velocity of $x \in \{\mathrm{dist}(\cdot, t) = s\}$ equals the normal velocity of its projection y on $\partial E(t)$, which is given by the anisotropic curvature $\kappa_\phi^{E(t)}(y)$ of $\partial E(t)$ at y. Since (as long as the surfaces are smooth)

$$\kappa_\phi^{\{d(\cdot,t)\le s\}}(x) = \mathrm{div}\,(\nabla\phi(\nabla d))(x, t) \le \kappa_\phi^{E(t)}(y),$$

we deduce that

$$\partial_t d \ge \mathrm{div}\,(\nabla\phi(\nabla d)) \qquad \text{in } \{d > 0\} \tag{4}$$

as long as $E(\cdot)$ is nonempty. In words, the positive level sets of the distance function shrink with a velocity which is higher than that given by its anisotropic curvature, and thus they may be regarded as superflows or *supersolutions* of the geometric motion. Analogously, setting $d^c(\cdot, t) := \mathrm{dist}(\cdot, E^c(t))$, where E^c stands for the complement of E, we have

$$\partial_t d^c \ge \mathrm{div}\,(\nabla\phi(\nabla d^c)) \qquad \text{in } \{d^c > 0\} \tag{5}$$

as long as $E^c(\cdot)$ is nonempty. We may conclude that a smooth flow $t \mapsto E(t)$ of sets solves (1) if and only if (4) and (5) are satisfied.

As already remarked before, when ϕ is crystalline $\nabla\phi(\nabla d)$ may not be defined and must be replaced in general by a suitable selection of the subdifferential map, that is, by a vector-field $z \in L^\infty(\{d > 0\}; \mathbb{R}^N)$ such that $z(x) \in \partial\phi(\nabla d(x))$ for a.e. x, where $\partial\phi$ denotes the subdifferential of ϕ. Any such z will be called *admissible for* d.

The above discussion motivates the following weak formulation of the crystalline flow: we will say that a one-parameter family $t \mapsto E(t)$ of closed sets, satisfying suitable continuity properties (see Definition 2.1 below) is a *weak supersolution* of (1) with initial datum E^0 if $E(0) \subseteq E^0$ and there exists a vector-field z, admissible for d, such that (4) hold in the sense of distributions, with $\nabla\phi(\nabla d)$ replaced by z. We will say instead that $t \mapsto E(t)$ is a weak-subsolution of (1) if $E(0) \supseteq E^0$ and $t \mapsto (\mathring{E}(t))^c$ is weak supersolution. Finally, we will say that $t \mapsto E(t)$ is a *weak solution* if it is both a weak sub- and a supersolution (with initial datum E^0). Mostly for technical reasons, we will require in addition that the positive part of $\operatorname{div} z$ is bounded in $\{d \geq \delta\}$ for all $\delta > 0$.

Let us notice that this formulation of the curvature flow in terms of the distance function has been already exploited for the standard mean curvature motion and its regular anisotropic variants. In fact, it is close in spirit to the distance formulation proposed and studied in [47], the main difference being that the differential inequalities are required to hold in a distributional sense, rather than in the viscosity sense. In this respect, our formulation is reminiscent of the approach developed in [15] where it was observed that the flat flow, in the convex case, was satisfying similar inequalities. In smooth cases, we can show it is more or less equivalent to the viscosity approach, *cf* Remark 2.3 and the Appendix.

We now describe the plan of the paper. In Section 2, after recalling some preliminaries definitions and introducing the main notation, we give the precise weak formulation of the sub- and supersolutions to the anisotropic mean curvature flow. In Section 3 we establish a comparison principle between sub- and supersolutions, which by standard arguments yields the uniqueness of the crystalline flow whenever fattening does not occur. We remark that the distributional formulation described above allows for a proof of the comparison, which is closer in spirit to the uniqueness proofs for standard parabolic equations. In particular, our argument is more elementary than the typical "viscosity" proof that is based on delicate regularization procedures and fine differentiability properties of semiconvex functions. In Section 4 we provide an existence result for the the weak formulation of the crystalline flow, which is based on the reformulation of the minimizing movements scheme of Almgren-Taylor-Wang / Luckhaus-Sturzenhecker introduced in [18, 15]. Such a variant can be considered as a combination of the ideas of [2] and the threshold dynamics algorithm studied in [26], and has several advantages: for instance, it makes it easier to establish a comparison principle for the discrete-in-time evolutions and it works equally well for bounded and unbounded sets (as already exploited in [24]). In the main theorem of the section (see Theorem 4.5) we establish the convergence of the minimizing movements scheme to a weak solution, whenever no fattening occurs.

We conclude this introduction by commenting on the restriction $m = \phi$ in (1). Although such a mobility is rather natural (for instance it forces Wulff shapes to evolve in a self-similar way), it is not the most general case and different mobilities could be considered as physically interesting. However, at the moment, in the crystalline case we are able to provide the right convergence estimates for the minimizing movements scheme only under this assumption; the main technical reason is related to the fact that if dist is the distance induced by the polar norm ϕ°, then the crystalline curvatures of the positive level sets of $\operatorname{dist}(\cdot, E)$ are bounded above (this can be easily understood since in this case the sublevel sets of $\operatorname{dist}(\cdot, E)$ admit an inner tangent Wulff shape at all points of the boundary). Nevertheless, we remark that in the case of a smooth elliptic anisotropy, all our results and methods would work with *any* mobility m, thus showing that the viscosity solutions already studied in [27, 21, 47] satisfy in fact a stronger (distributional) formulation. The extension of our results to more general mobilities in the crystalline case will be the subject of future investigations.

2 A weak formulation of the crystalline mean curvature flow

In this section we introduce a suitable weak formulation of the crystalline mean curvature flow. Such a notion of solution resembles the formulation due to [47]. However, here we will not consider the viscosity setting of [47] and we will rather be concerned with distributional solutions (which appear for instance in [15]).

2.1 Preliminaries

In this subsection we introduce the main objects and notation used throughout the paper.

Let ϕ denote a fixed norm on $\mathbb{R}^N$, that is, a convex, even and 1-homogeneous real-valued function, which will play the role of the anisotropic interfacial energy density. In the terminology of crystal growth this is also called *surface tension*. Note that we do not assume any further regularity on ϕ and in fact the main case of interest is when ϕ is *crystalline*, that is, when the associated unit ball is a convex polytope. The interfacial energy is then given by

$$P_\phi(E) := \sup\left\{ \int_E \operatorname{div}\zeta\, dx : \zeta \in C^1_c(\mathbb{R}^N;\mathbb{R}^N),\ \phi^\circ(\zeta) \le 1 \right\},$$

where we recall that the *polar norm* ϕ° is defined as

$$\phi^\circ(\xi) := \sup_{\phi(\eta)\le 1} \eta \cdot \xi\,. \tag{6}$$

It can be checked that $P_\phi(E)$ is finite if and only if E is a set of finite perimeter and, in this case,

$$P_\phi(E) = \int_{\partial^* E} \phi(\nu^E)\, d\mathcal{H}^{N-1}\,,$$

where $\partial^* E$ denotes the so-called reduced boundary of E (see for instance [5]). More generally, given a function $u \in BV_{loc}(\mathbb{R}^N)$ we may consider the *anisotropic total variation maesure* of u, which on the open (bounded if $u \notin BV(\mathbb{R}^N)$) subsets $\Omega \subset \mathbb{R}^N$ is defined as

$$\phi(Du)(\Omega) := \sup\left\{ \int_\Omega u \operatorname{div}\zeta\, dx : \zeta \in C^1_c(\Omega;\mathbb{R}^N),\ \phi^\circ(\zeta) \le 1 \right\}.$$

Because of the homogeneity of ϕ it turns out that $\phi(Du)$ coincides with the non-negative Radon measure in $\mathbb{R}^N$ given by $\phi(\nabla u)\, dx + \phi\left(\frac{D^s u}{|D^s u|}\right) |D^s u|$, where ∇u

stands for the absolutely continuous part of Du and $\frac{D^s u}{|D^s u|}$ denotes the Radon-Nykodim derivative of the singular part $D^s u$ of Du with respect to its (isotropic) total variation $|D^s u|$, see [5].

Among the important properties of ϕ and ϕ° let us mention the fact that $\partial\phi(0) = \{\xi : \phi^\circ(\xi) \le 1\}$ while $\partial\phi^\circ(0) = \{\xi : \phi(\xi) \le 1\}$. Moreover, for $\eta \neq 0$

$$\partial\phi(\eta) = \{\xi : \phi^\circ(\xi) \le 1 \text{ and } \xi\cdot\eta = \phi(\eta)\} = \{\xi : \phi^\circ(\xi) = 1 \text{ and } \xi\cdot\eta = \phi(\eta)\} \quad (7)$$

(and the symmetric statement for ϕ°). An easy consequence of the above characterization is that if $\eta \in \partial\phi^\circ(x)$ and $x \neq 0$, then $x/\phi^\circ(x) \in \partial\phi(\eta)$.

The set

$$W(0,1) := \{y : \phi^\circ(y) \le 1\}$$

is called the *Wulff shape* associated with ϕ. More generally, for $x \in \mathbb{R}^N$ and $R > 0$, we will denote by

$$W(x,R) := \{y : \phi^\circ(y-x) \le R\}$$

the *Wulff shape of radius R and center x*. In the Finsler metric framework associated with ϕ°, Wulff shapes play the same role as standard balls do in the Euclidean setting. In particular, it is well-known that $W(0,R)$ is the unique (up to translations) solution of the anisotropic isoperimetric problem

$$\min\left\{P_\phi(E) : \, |E| = |W(0,R)|\right\},$$

see for instance [29].

Given a set $E \subseteq \mathbb{R}^N$, we denote by $\mathrm{dist}(\cdot, E)$ the distance from E induced by ϕ°, that is, for any $x \in \mathbb{R}^N$

$$\mathrm{dist}(x,E) := \inf_{y\in E} \phi^\circ(x-y) \quad (8)$$

if $E \neq \emptyset$ and $\mathrm{dist}(x,\emptyset) := +\infty$. Moreover, we denote by d_E the signed distance from E induced by ϕ°, i.e.,

$$d_E(x) := \mathrm{dist}(x,E) - \mathrm{dist}(x,E^c)$$

so that $\mathrm{dist}(x,E) = d_E(x)^+$ and $\mathrm{dist}(x,E^c) = d_E(x)^-$ (here and throughout the paper we adopt the standard notation $t^+ := t \vee 0$ and $t^- := (-t)^+$). Note that $\phi(\nabla d_E) = 1$ a.e. in $\mathbb{R}^N \setminus \partial E$.

We finally recall the notion of Kuratowski convergence. We say that a sequence of closed sets E_n in $\mathbb{R}^m$ converges to a closed set E in the Kuratowki sense, and we write

$$E_n \xrightarrow{\mathcal{K}} E,$$

if the following conditions are satisfied:

(i) if $x_n \in E_n$, any limit point of $\{x_n\}$ belongs to E;

(ii) any $x \in E$ is the limit of a sequence $\{x_n\}$, with $x_n \in E_n$.

One can easily see that $E_n \xrightarrow{\mathcal{K}} E$ if and only if $\mathrm{dist}(\cdot, E_n) \to \mathrm{dist}(\cdot, E)$ locally uniformly in $\mathbb{R}^m$ (here one may consider the distance associated to any norm). In particular, by the Ascoli-Arzelà Theorem, any sequence of closed sets admits a subsequence which converges in the Kuratowski sense.

2.2 A weak formulation of the crystalline flow

In this subsection we introduce the weak formulation of the crystalline flow we will deal with. We refer the reader to the introduction for the motivation behind this definition.

Definition 2.1. Let $E^0 \subset \mathbb{R}^N$ be a closed set. Let E be a closed set in $\mathbb{R}^N \times [0, +\infty)$ and for each $t \geq 0$ denote $E(t) := \{x \in \mathbb{R}^N : (x,t) \in E\}$. We say that E is a *supersolution* of the curvature flow (1) with initial datum E^0 if

(a) $E(0) \subseteq E^0$;

(b) for all $t \geq 0$ if $E(t) = \emptyset$, then $E(s) = \emptyset$ for all $s > t$;

(c) $E(s) \xrightarrow{\mathcal{K}} E(t)$ as $s \nearrow t$ for all $t > 0$ (left-continuity);

(d) setting $d(x,t) := \operatorname{dist}(x, E(t))$ for $(x,t) \in \mathbb{R}^N \times (0, T^*) \setminus E$ and

$$T^* := \inf\{t > 0 : E(s) = \emptyset \text{ for } s \geq t\},$$

then the inequality

$$\partial_t d \geq \operatorname{div} z \tag{9}$$

holds in the distributional sense in $\mathbb{R}^N \times (0, T^*) \setminus E$ for a suitable $z \in L^\infty(\mathbb{R}^N \times (0,T^*); \mathbb{R}^N)$ such that $z \in \partial\phi(\nabla d)$ a.e., $\operatorname{div} z$ is a Radon measure in $\mathbb{R}^N \times (0,T^*) \setminus E$, and $(\operatorname{div} z)^+ \in L^\infty(\{(x,t) \in \mathbb{R}^N \times (0,T^*) : d(x,t) \geq \delta\})$ for every $\delta > 0$.

We say that A, open set in $\mathbb{R}^N \times [0, +\infty)$, is a subsolution with initial datum E^0 if A^c is a supersolution with initial datum $(\mathring{E}^0)^c$.

Finally, we say that E, closed set in $\mathbb{R}^N \times [0, +\infty)$, is a solution with initial datum E^0 if it is a supersolution and if $\mathring{E}$ is a subsolution, both with initial datum E^0.

Remark 2.2. Notice that the initial condition for subsolutions may be rewritten as $\mathring{E}^0 \subseteq A(0)$. In particular, if $\partial E^0 = \partial \mathring{E}^0$ and E is a solution according to the previous definition, then $E(0) = E^0$.

Remark 2.3. If ϕ is C^2, then one can check that this definition is stronger than the definition in the viscosity sense (see in particular [47, 10]). If in addition ϕ^2 is strongly convex, this is an equivalence, see Proposition A.1 in the Appendix.

We start by observing some useful continuity properties of the map d introduced in the previous definition.

Lemma 2.4. *Let E be a supersolution. Then, for each $t \in [0, T^*)$, $d(\cdot, s)$ converges locally uniformly in $\{x : d(x,t) > 0\}$ as $s \searrow t$ to for some function d^r with $d^r \geq d(\cdot, t)$ in $\{x : d(x,t) > 0\}$.*

Proof. By condition (d) of Definition 2.1, the distributional derivative $\partial_t d$ is a Radon measure in $\mathbb{R}^N \times (0, T^*) \setminus E$, so that d is locally a function with bounded variation in this (open) domain. In particular, for a.e. $x \in \mathbb{R}^N$ the map

$s \mapsto d(x,s)$ has a right limit $d^r(x,t)$ at each time $t \in [0,T^*)$ such that $d(x,t) > 0$. Since the functions $d(\cdot,s)$ are also equi-Lipschitz in space as s varies, we may conclude that the right limit is in fact locally uniform in $\{x : d(x,t) > 0\}$.

Since E is closed, for every $t \in [0,T^*]$ we clearly have that all Kuratowski cluster points of $E(s)$ as $s \to t$ are contained in $E(t)$. In other words, one has $d(x,t) \leq \liminf_{s\to t} d(x,s)$. Thus, $d^r \geq d(\cdot,t)$ in $\{x : d(x,t) > 0\}$. □

Remark 2.5. Observe that by condition (c) in the definition (which is mostly technical and forbids artificial constructions such as a supersolution which jumps to $E(t) = \mathbb{R}^N$ at a given time $t > 0$), $t \mapsto d(\cdot,t) := d(\cdot,E(t))$ is left-continuous with respect to the local uniform convergence.

3 Comparison results

In this section we prove the main comparison principle between sub- and supersolutions (see Theorem 3.3). In Lemma 3.2 below, we establish a first (suboptimal) comparison result between a supersolution and a suitable anisotropic total variation flow (see [11, 44]). To this aim, we give an explicit solution to the anisotropic total variation flow with initial datum ϕ°.

Lemma 3.1. *The pair (f,ζ) defined by*

$$f(x,t) := \begin{cases} r(t) + t\frac{N-1}{r(t)} & \text{if } \phi^\circ(x) \leq r(t) := \sqrt{(N+1)t}, \\ \phi^\circ(x) + t\frac{N-1}{\phi^\circ(x)} & \text{otherwise} \end{cases} \tag{10}$$

and

$$\zeta(x,t) := \begin{cases} \frac{x}{r(t)} & \text{if } \phi^\circ(x) \leq r(t), \\ \frac{x}{\phi^\circ(x)} & \text{if } \phi^\circ(x) \geq r(t), \end{cases} \tag{11}$$

solve the following Cauchy problem for the ϕ-total variation flow in $\mathbb{R}^N$:

$$\begin{cases} \partial_t f = \operatorname{div} \zeta & \text{a.e. in } \mathbb{R}^N \times (0,+\infty), \\ \zeta \in \partial\phi(\nabla f) & \text{a.e. in } \mathbb{R}^N \times (0,+\infty), \\ f(\cdot,0) = \phi^\circ. \end{cases} \tag{12}$$

Moreover, given $\lambda > 1$, the pair $(f_\lambda, \zeta_\lambda)$ given by

$$f_\lambda(x,t) := \lambda f(x,t/\lambda) \qquad \zeta_\lambda(x,t) := \zeta(x,t/\lambda)$$

for $(x,t) \in \mathbb{R}^N \times (0,+\infty)$ solves (12)*, with the initial datum ϕ° replaced by $\lambda\phi^\circ$.*

Proof. Recalling that $\zeta \in \partial\phi(\nabla f)$ is equivalent to $\phi^\circ(\zeta) \leq 1$, $\zeta \cdot \nabla f = \phi(\nabla f)$ (see (7)), the proof follows by direct verification. The details are left to the reader. □

Next lemma provides a first comparison estimate, which is far from being sharp. However, the optimal estimate can be established a posteriori as a consequence of our main comparison theorem (see Theorem 3.3 below).

Lemma 3.2. *Let E be a supersolution and $d := \mathrm{dist}(\cdot, E(\cdot))$ the associated one parameter family of distance functions. Assume that for some $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times [0, +\infty)$ we have $d(\bar{x}, \bar{t}) \geq R > 0$. Then, there exists a constant $\chi_N > 0$ such that $d(\bar{x}, \bar{t} + s) \geq R - \chi_N \sqrt{s}$ for all $s \in [0, R^2/(16\chi_N^2)]$.*

Proof. Observe first that thanks to Lemma 2.4, since $d(\cdot, \bar{t}) \geq R/4$ in $\{x : \phi^\circ(x - \bar{x}) \leq 3R/4\} = W(\bar{x}, 3R/4)$, there exists a (unknown) time t^* such that $d(\cdot, \bar{t} + s) > \alpha > 0$ in $W(\bar{x}, 3R/4)$ for all $s \in [0, t^*]$ for some positive α. We will compare d with the solution δ of the ϕ-total variation flow starting from

$$\delta(\cdot, 0) := R - \frac{4}{3}\phi^\circ(\cdot - \bar{x}) .$$

More precisely, if we introduce $\delta(x, s) := R - f_{4/3}(x - \bar{x}, s)$, where for any $\beta > 0$, $f_\beta(x, t) := \beta f(x, t/\beta)$ and f is given by (10), by Lemma 3.1 the function δ satisfies

$$\begin{cases} \partial_t \delta = \operatorname{div} \xi & \text{in } \mathbb{R}^N \times (0, +\infty), \\ \xi \in \partial\phi(\nabla\delta) & \text{a.e. in } \mathbb{R}^N \times (0, +\infty), \end{cases} \tag{13}$$

where $\xi(x, t) = -\zeta(x, 3t/4)$, with ζ defined by (11). Note that δ is negative outside $W(\bar{x}, 3R/4)$ for all positive times.

Let $\Psi(s)$ be a smooth, convex, nonnegative function, which vanishes only for $s \leq 0$, and consider the function $w(x, s) := \Psi(\delta(x, s) - d(x, \bar{t} + s))$. Without loss of generality, we assume to simplify the notation that $\bar{t} = 0$. By construction, $w(x, 0) \equiv 0$ in $W(\bar{x}, 3R/4)$ and $w(\cdot, s) \equiv 0$ on $\partial W(\bar{x}, 3R/4)$ for $0 \leq s \leq t^*$.

Since $\phi(\nabla d) \leq 1$ a.e. and $\partial_t d$ is a measure wherever d is positive, it follows that d is a function in $BV_{loc}(W(\bar{x}, 3R/4) \times (0, t^*))$ and its distributional time derivative has the form

$$\partial_t d = \sum_{t \in J} [d(\cdot, t + 0) - d(\cdot, t - 0)] dx + \partial_t^d d$$

where J is the (countable) set of times where d jumps and $\partial_t^d d$ is the diffuse (Cantor+absolutely continuous) part of the derivative. It turns out that $d(\cdot, t + 0) - d(\cdot, t - 0) \geq 0$ for each $t \in J$ (*cf* Lemma 2.4). Moreover, since the positive part of $\operatorname{div} z$ is absolutely continuous with respect to the Lebesque measure, (9) entails

$$\partial_t^d d \geq \operatorname{div} z.$$

Using the chain rule for BV functions, see [4]), one has

$$\begin{aligned} \partial_t w = \sum_{t \in J} [\Psi(\delta(\cdot, t) - d(\cdot, t + 0)) - \Psi(\delta(\cdot, t) - d(\cdot, t - 0))] dx \\ + \Psi'(\delta - d)(\partial_t \delta - \partial_t^d d) \leq \Psi'(\delta - d)(\operatorname{div} \xi - \operatorname{div} z). \end{aligned}$$

Hence, for a.e. $t \leq t^*$, using the fact that ϕ and Ψ are convex, $\Psi'(\delta - d)$ vanishes on $\partial W(\bar{x}, 3R/4)$ and recalling (13), we have

$$\partial_t \int_{W(\bar{x},3R/4)} w dx \leq \int_{W(\bar{x},3R/4)} \Psi'(\delta - d)(\operatorname{div} \xi - \operatorname{div} z)$$
$$= - \int_{W(\bar{x},3R/4)} (\xi - z) \cdot (\nabla \delta - \nabla d)\Psi''(\delta - d) \leq 0.$$

It follows that $w = \Psi(\delta - d) = 0$, that is, $d \geq \delta$ a.e. at all times less than t^*. More precisely, for $0 \leq s \leq t^*$ we have

$$d(\bar{x}, \bar{t} + s) \geq R - f_{4/3}(x - \bar{x}, s) = R - \frac{4N}{\sqrt{3}}\sqrt{\frac{s}{N+1}} =: R - \chi_N \sqrt{s}. \tag{14}$$

It follows that $d(\bar{x}, \bar{t} + s) > 3R/4$ and, in turn, $d(\cdot, \bar{t} + s) > 0$ on $\partial W(\bar{x}, 3R/4)$ for all $s < \min\{t^*, R^2/(16\chi_N^2)\}$. But then we can restart the argument above to find that (14) remains valid for slightly larger times. Thus, we may conclude that (14) holds at least for all $0 \leq s \leq R^2/(16\chi_N^2)$. This concludes the proof of the lemma.

□

Now we can state the main result of this section, which is a comparison result between sub- and supersolutions.

Theorem 3.3. *Let E be a supersolution with initial datum E^0 and F be a subsolution with initial datum F^0. Assume that* $\operatorname{dist}(E^0, {F^0}^c) =: \Delta > 0$*. Then for each $t \geq 0$,* $\operatorname{dist}(E(t), F^c(t)) \geq \Delta$*.*

Proof. Let T_E^* and T_F^* be the maximal existence time for E and F. For all $t > \min\{T_E^*, T_F^*\}$ we have that either E or F^c is empty. In this case, clearly the conclusion holds true.

Now, consider the case $t \leq \min\{T_E^*, T_F^*\}$ (and assume without loss of generality that $T_E^*, T_F^* > 0$). Let us fix $0 < \eta_1 < \eta_1' < \eta_1'' < \eta_2'' < \eta_2' < \eta_2 < \Delta$. We will show the conclusion of the theorem for a time interval $(0, t^*)$ for a suitable t^* depending only on η_1, η_1', η_1'', η_2'', η_2', η_2, and ultimately only on Δ. It is clear then that reiterating the argument yields the conclusion of the theorem for all times. We recall that $d_E(x, t) := \operatorname{dist}(x, E(t)) - \operatorname{dist}(x, E^c(t))$ and d_F is defined analogously. We denote by z_E and z_F the fields appearing in the definition of super- and subsolutions (see Definition 2.1), corresponding to E and F, respectively. Define

$$S := \{x \in \mathbb{R}^N : \eta_1 < d_E(x, 0) < \eta_2\}$$

and note that by Lemma 3.2 there exists $t^* > 0$ depending only on η_1, $\Delta - \eta_2$ such that

$$\begin{aligned} d_E(x,t) &\geq d_E(x,0) - \chi_N \sqrt{t} \\ d_F(x,t) &\leq d_F(x,0) + \chi_N \sqrt{t} \end{aligned} \qquad \text{for all } x \in \overline{S} \text{ and } t \in (0, t^*). \tag{15}$$

We now set

$$\tilde{d}_E := d_E \vee (\eta_1' + \chi_N \sqrt{t}) \,,$$
$$\tilde{d}_F := (d_F + \Delta) \wedge (\eta_2' - \chi_N \sqrt{t}) \,.$$

Clearly, by our assumptions $\tilde{d}_E(\cdot, 0) \geq \tilde{d}_F(\cdot, 0)$. We claim that

$$\tilde{d}_E \geq \tilde{d}_F \text{ on } \partial S \times (0, t^*) \,. \tag{16}$$

Here and in the rest of the proof we may assume without loss of generality that t^* is as small as needed (but still depending only on Δ). To this aim, write $\partial S = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 := \{d_E(\cdot, 0) = \eta_1\}$ and $\Gamma_2 := \{d_E(\cdot, 0) = \eta_2\}$. Since $d_F(\cdot, 0) + \Delta \leq d_E(\cdot, 0) = \eta_1$ on Γ_1, we deduce

$$\tilde{d}_F \leq d_F + \Delta \leq \eta_1 + \chi_N \sqrt{t} \leq \eta_1' \leq \tilde{d}_E$$

on $\Gamma_1 \times (0, t^*)$. Similarly one can show that the inequality $\tilde{d}_E \geq \tilde{d}_F$ holds on $\Gamma_2 \times (0, t^*)$.

Again by (15) we have

$$d_E \geq \frac{\eta_1''}{2} > 0 \quad \text{in } \{d_E(\cdot, 0) \geq \eta_1''\} \times (0, t^*) \tag{17}$$

and, observing that $d_F(\cdot, 0) \leq \eta_2'' - \Delta$ in $\{d_E(\cdot, 0) \leq \eta_2''\}$,

$$d_F \leq \frac{\eta_2'' - \Delta}{2} < 0 \quad \text{in } \{d_E(\cdot, 0) \leq \eta_2''\} \times (0, t^*) \,. \tag{18}$$

In particular

$$E(t) \subset\subset F(t) \qquad \text{for } t \in (0, t^*) \,.$$

We now claim that, setting

$$S'' := \{x \in \mathbb{R}^N : \eta_1'' < d_E(x, 0) < \eta_2''\},$$

we have

$$\tilde{d}_E = d_E \quad \text{and} \quad \tilde{d}_F = d_F + \Delta \qquad \text{in } S'' \times (0, t^*) \,. \tag{19}$$

Indeed by (15) we have

$$d_E(x, t) \geq \eta_1'' - \chi_N \sqrt{t} \geq \eta_1' + \chi_N \sqrt{t} \quad \text{for } (x, t) \in S'' \times (0, t^*)$$

and thus $\tilde{d}_E = d_E$ in $S'' \times (0, t^*)$. The proof of the second identity in (19) is analogous.

Now we will use quite standard parabolic maximum principles, like in the proof of Lemma 3.2. Notice that

$$\partial_t \tilde{d}_E = \sum_{t \in J} [\tilde{d}_E(\cdot, t + 0) - \tilde{d}_E(\cdot, t - 0)] dx + \partial_t^d \tilde{d}_E \,,$$

where J is the (countable) set of times where d_E possibly jumps and $\partial_t^d \tilde{d}_E$ is the diffuse part of the distributional derivative. Using for instance the chain rule proved in [4], in $S \times (0, t^*)$ we have that

$$\partial_t^d \tilde{d}_E = \begin{cases} \frac{\chi_N}{2\sqrt{t}} & \text{a.e. in } \{(x,t) : \eta_1' + \chi_N \sqrt{t} > d_E(x,t)\}\,, \\ \partial_t^d d_E & |\partial_t^d d_E|\text{-a.e. in } \{(x,t) : \eta_1' + \chi_N \sqrt{t} \le d_E(x,t)\}\,. \end{cases}$$

An analogous formula holds for $\partial_t^d \tilde{d}_F$. Recalling that $(\operatorname{div} z_E)^+$ and $(\operatorname{div} z_F)^-$ belong to $L^\infty(S \times (0,t^*))$ it follows that (possibly modifying t^*)

$$\partial_t^d \tilde{d}_E \ge \operatorname{div} z_E \quad \text{and} \quad \partial_t^d \tilde{d}_F \le \operatorname{div} z_F \tag{20}$$

in the sense of measures in $S \times (0, t^*)$. Note also that a.e. in $S \times (0, t^*)$

$$z_E \in \partial\phi(\nabla \tilde{d}_E) \quad \text{and} \quad z_F \in \partial\phi(\nabla \tilde{d}_F)\,. \tag{21}$$

Fix $p > N$ and set $\Psi(s) := (s^+)^p$ and $w := \Psi(\tilde{d}_F - \tilde{d}_E)$. By (16) we have

$$w = 0 \qquad \text{on } \partial S \times (0, t^*)\,. \tag{22}$$

Using as before the chain rule for BV functions, recalling (20) and the fact that the jump parts of $\partial_t \tilde{d}_E$ and $\partial_t \tilde{d}_F$ are nonnegative and nonpositive, respectively, we have

$$\partial_t w \le \Psi'(\tilde{d}_F - \tilde{d}_E)(\partial_t^d \tilde{d}_F - \partial_t^d \tilde{d}_E) \le \Psi'(\tilde{d}_F - \tilde{d}_E)(\operatorname{div} z_F - \operatorname{div} z_E) \tag{23}$$

in $S \times (0, t^*)$. Choose a cut-off function $\eta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on B_1. For every $\varepsilon > 0$ we set $\eta_\varepsilon(x) := \eta(\varepsilon x)$. Using (22) and (23), we have

$$\begin{aligned} \partial_t \int_S w \eta_\varepsilon^p dx \le & \int_S \eta_\varepsilon^p \Psi'(\tilde{d}_F - \tilde{d}_E)(\operatorname{div} z_F - \operatorname{div} z_E) \\ & = - \int_S \eta_\varepsilon^p \Psi''(\tilde{d}_F - \tilde{d}_E)(z_F - z_E) \cdot (\nabla \tilde{d}_F - \nabla \tilde{d}_E)\, dx + \\ & \quad p \int_S \eta_\varepsilon^{p-1}\, \Psi'(\tilde{d}_F - \tilde{d}_E) \nabla \eta_\varepsilon \cdot (z_F - z_E)\, dx \\ & \le p \int_S \eta_\varepsilon^{p-1}\, \Psi'(\tilde{d}_F - \tilde{d}_E) \nabla \eta_\varepsilon \cdot (z_F - z_E)\, dx, \end{aligned}$$

where we have also used the inequality $(z_F - z_E) \cdot (\nabla \tilde{d}_F - \nabla \tilde{d}_E) \ge 0$, which follows from (21) and the convexity of ϕ. By Hölder Inequality and using the explicit expression of Ψ and Ψ', we get

$$\partial_t \int_S w\, \eta_\varepsilon^p dx \le C p^2 \|\nabla \eta_\varepsilon\|_{L^p(\mathbb{R}^N)} \left(\int_S w\, \eta_\varepsilon^p dx \right)^{1 - \frac{1}{p}},$$

for some constant $C > 0$ depending only on the L^∞-norms of z_E and z_F. Since $w = 0$ at $t = 0$, a simple ODE argument then yields

$$\int_S w\, \eta_\varepsilon^p dx \le \left(C p \|\nabla \eta_\varepsilon\|_{L^p(\mathbb{R}^N)} t \right)^p$$

for all $t \in (0, t^*)$. Observing that $\|\nabla \eta_\varepsilon\|^p_{L^p(\mathbb{R}^N)} = \varepsilon^{p-N}\|\nabla \eta\|^p_{L^p(\mathbb{R}^N)} \to 0$ and $\eta_\varepsilon \nearrow 1$ as $\varepsilon \to 0^+$, we conclude that $w = 0$, and in turn $\tilde{d}_E \geq \tilde{d}_F$ in $S \times (0, t^*)$. In particular, by claim (19), we have shown that $d_E \geq d_F + \Delta$ in $S'' \times (0, t^*)$. We finally claim that $\mathrm{dist}(E(t), F^c(t)) \geq \Delta$ for $t \in (0, t^*)$. To see this, fix $\varepsilon \geq 0$, and let let $x \in \partial E(t)$ and $y \in \partial F(t)$ be such that $\phi^\circ(x - y) \leq \mathrm{dist}(E(t), F^c(t)) + \varepsilon$. Note that by (17) and (18) we have $d_E(x, 0) < \eta_1''$ and $d_E(y, 0) > \eta_2''$. Thus there exists $z \in S'' \cap [x, y]$, where $[x, y]$ denotes the segment joining x and y. Since $d_E(\cdot, t) \geq d_F(\cdot, t) + \Delta$ in S'', we have

$$\begin{aligned} \mathrm{dist}(E(t), F^c(t)) \geq \phi^\circ(x - y) - \varepsilon = \phi^\circ(x - z) + \phi^\circ(z - y) - \varepsilon \geq \\ - d_F(z, t) + d_E(z, t) - \varepsilon \geq \Delta - \varepsilon. \quad (24) \end{aligned}$$

The claim follows by the arbitrariness of ε, and this concludes the proof of the theorem. □

4 Existence via minimizing movements

In this section we prove an existence result for the crystalline curvature flow, according to Definition 2.1. Such a solution is obtained via a variant of the Almgren-Taylor-Wang minimizing movements scheme ([2]) introduced in [18, 15, 16].

4.1 Minimizing movements

Let $E^0 \subset \mathbb{R}^N$ be closed. Fix a time-step $h > 0$ and set $E_h^0 = E^0$. We then inductively define E_h^{k+1} (for all $k \in \mathbb{N}$) according to the following procedure: If $E_h^k \neq \emptyset, \mathbb{R}^N$, then let $(u_h^{k+1}, z_h^{k+1}) : \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}^N$ satisfy

$$\begin{cases} -h \,\mathrm{div}\, z_h^{k+1} + u_h^{k+1} = d_{E_h^k}, \\ z_h^{k+1} \in \partial\phi(\nabla u_h^{k+1}) \quad \text{a.e. in } \mathbb{R}^N, \end{cases} \tag{25}$$

and set $E_h^{k+1} := \{x : u_h^{k+1} \leq 0\}$. If either $E_h^k = \emptyset$ or $E_h^k = \mathbb{R}^N$, then set $E_h^{k+1} := E_h^k$. We denote by T_h^* the first discrete time hk such that $E_h^k = \emptyset$, if such a time exists; otherwise we set $T_h^* = +\infty$. Analogously, we denote by ${T'_h}^*$ the first discrete time hk such that $E_h^k = \mathbb{R}^N$, if such a time exists; otherwise we set ${T'_h}^* = +\infty$. In Proposition 4.1 below we will show that this construction is well defined, since problem (25) admits a unique solution u_h^{k+1} that is Lipschitz continuous. In particular, E_h^{k+1} is a closed set for all k.

Before stating the main facts about the differential problem (25), we recall that given $z \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ with $\mathrm{div}\, z \in L^2_{loc}(\mathbb{R}^N)$ and $w \in BV_{loc}(\mathbb{R}^N) \cap L^2_{loc}(\mathbb{R}^N)$, $z \cdot Dw$ denotes the Radon measure associated with the linear functional

$$L\varphi := -\int_{\mathbb{R}^N} w\,\varphi \mathrm{div}\, z\, dx - \int_{\mathbb{R}^N} w\, z \cdot \nabla\varphi\, dx \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N),$$

see [9].

Proposition 4.1. *Let $g \in L^2_{loc}(\mathbb{R}^N)$. There exists a field $z \in L^\infty(\mathbb{R}^N; W(0,1))$ and a unique function $u \in BV_{loc}(\mathbb{R}^N) \cap L^2_{loc}(\mathbb{R}^N)$ such that the pair (u,z) satisfies*

$$\begin{cases} -h \operatorname{div} z + u = g & \textit{in } \mathcal{D}'(\mathbb{R}^N), \\ \phi^\circ(z) \le 1 & \textit{a.e. in } \mathbb{R}^N, \\ z \cdot Du = \phi(Du) & \textit{in the sense of measures.} \end{cases} \tag{26}$$

Moreover, for any $R > 0$ and $v \in BV(B_R)$ with $\operatorname{Supp}(u-v) \Subset B_R$,

$$\phi(Du)(B_R) + \frac{1}{2h}\int_{B_R} (u-g)^2\,dx \le \phi(Dv)(B_R) + \frac{1}{2h}\int_{B_R} (v-g)^2\,dx,$$

and for every $s \in \mathbb{R}$ the set $E_s := \{x \in \mathbb{R}^N : u(x) \le s\}$ solves the minimization problem

$$\min_{F \Delta E_s \Subset B_R} P_\phi(F; B_R) + \frac{1}{h}\int_{F \cap B_R} (g(x) - s)\,dx.$$

If $g_1 \le g_2$ and if u_1, u_2 are the corresponding solutions to (26) *(with g replaced by g_1 and g_2, respectively), then $u_1 \le u_2$.*

Finally if in addition g is Lipschitz with $\phi(\nabla g) \le 1$, then the unique solution u of (26) *is also Lipschitz and satisfies $\phi(\nabla u) \le 1$ a.e. in $\mathbb{R}^N$. As a consequence,* (26) *is equivalent to*

$$\begin{cases} -h \operatorname{div} z + u = g & \textit{in } \mathcal{D}'(\mathbb{R}^N)\,, \\ z \in \partial\phi(\nabla u) & \textit{a.e. in } \mathbb{R}^N \end{cases} \tag{27}$$

Proof. See [15, Theorem 2], [1, Theorem 3.3]. □

Remark 4.2 (Consistency with the ATW scheme). When ∂E^0 is bounded, the minimality property of the level sets stated above shows, in particular, that the sets E_h^k are constructed according to the Almgren-Taylor-Wang scheme [2].

Since by the previous proposition $\phi(\nabla u_h^{k+1}) \le 1$ a.e. in $\mathbb{R}^N$, one deduces, in particular, that

$$\begin{array}{ll} u_h^{k+1} \le d_{E_h^{k+1}} & \text{in } \{x : \operatorname{dist}(x, E_h^{k+1}) > 0\}\,, \\ u_h^{k+1} \ge d_{E_h^{k+1}} & \text{in } \{x : \operatorname{dist}(x, E_h^{k+1}) < 0\}\,. \end{array} \tag{28}$$

We are now in a position to define the time discrete evolutions. Precisely, we set

$$\begin{aligned} & E_h := \{(x,t) : x \in E_h^{[t/h]}\}, \\ & E_h(t) := E_h^{[t/h]} = \{x : (x,t) \in E_h\}, \\ & d_h(x,t) := d_{E_h(t)}(x), \\ & u_h(x,t) := u_h^{[t/h]}(x), \\ & z_h(x,t) := z_h^{[t/h]}(x), \end{aligned} \tag{29}$$

where $[\cdot]$ stands for the integer part of its argument.

Remark 4.3 (Discrete comparison principle). The last part of Proposition 4.1 implies that the scheme is monotone, that is, the discrete evolutions satisfy the comparison principle. More precisely, if $E^0 \subseteq F^0$ are closed sets and if we denote by E_h and F_h the discrete evolutions with initial datum E^0 and F^0, respectively, then $E_h(t) \subseteq F_h(t)$ for all $t \geq 0$. Analogously, if $E^0 \subset \overline{(F^0)^c}$, then $E_h(t) \subset \overline{(F_h(t))^c}$ for all $t \geq 0$.

4.2 Comparison with the Wulff shape

In this subsection, we exploit Remark 4.3 to compare the discrete evolutions (29) with the minimizing movements of the Wulff shape and derive an estimate, which will be useful in the convergence analysis. The evolution starting from a Wulff shape $W(0,R)$ is explicitly known. Indeed, from [15, Appendix B, Eq. (39)], the solution of (26), with g replaced by $d_{W(0,R)} = \phi^\circ - R$, is given by $(\phi^\circ_h - R, z_h)$, where

$$\phi^\circ_h(x) := \begin{cases} \sqrt{h}\frac{2N}{\sqrt{N+1}} & \text{if } \phi^\circ(x) \leq \sqrt{h(N+1)}, \\ \phi^\circ(x) + h\frac{N-1}{\phi^\circ(x)} & \text{otherwise,} \end{cases} \tag{30}$$

and

$$z_h(x) := \begin{cases} \frac{\left(2\sqrt{h(N+1)} - \phi^\circ(x)\right)x}{h(N+1)} & \text{if } \phi^\circ(x) \leq \sqrt{h(N+1)}, \\ \frac{x}{\phi^\circ(x)} & \text{otherwise.} \end{cases}$$

This can be checked by direct computation. It follows that if $E^0 = W(0,R)$, one has $E_h(t) = W(0, r^R_h(t))$ for a function r^R_h that satisfies

$$r^R_h(h) = \frac{R + \sqrt{R^2 - 4h(N-1)}}{2}$$

if $h \leq R^2/(4(N+1))$. In particular,

$$r^R_h(h) \geq \sqrt{R^2 - 4h(N-1)}$$

for the same h's. By iteration, we have $r^R_h(t) \geq \sqrt{R^2 - 4t(N-1)} \geq \frac{R}{\sqrt{2}}$ for $0 \leq t \leq R^2/(8(N-1))$ and $h \leq R^2/(8(N+1))$. Since $r^R_h(t) = R$ for $t \in [0,h)$, we infer

$$r^R_h(t) \geq \sqrt{R^2 - 4t(N-1)} \tag{31}$$

for $0 \leq t \leq R^2/(8(N+1))$ and for all h.

Now we return to the motion from an arbitrary set E^0. If for some $(x,t) \in \mathbb{R}^N \times [0, T'^*_h)$ we have $d_h(x,t) > R$, then $W(x,R) \cap E_h(t) = \emptyset$. Hence, by the comparison principle stated in Remark 4.3 and by (31) we have

$$d_h(x,s) \geq \sqrt{R^2 - 4(N-1)(s-t+h)}$$

for $t < s$ and $s + h - t < R^2/(8(N+1))$.

By letting $R \nearrow d_h(x,t)$ we obtain

$$d_h(x,s) \geq \sqrt{d^2_h(x,t) - 4(N-1)(s-t+h)} \tag{32}$$

for $t < s$ and $s + h - t < d_h^2(x,t)/(8(N+1))$.

By the same argument, if $d_h(x,t) < -R$ for some $(x,t) \in \mathbb{R}^N \times [0, T_h^*)$, then $W(x,R) \subset E_h(t)$ and thus, again by the discrete comparison principle and by (31) we have

$$d_h(x,s) \le -\sqrt{R^2 - 4(N-1)(s-t+h)}$$

for $t < s$ and $s + h - t < R^2/(8(N+1))$. Letting $R \nearrow -d_h(x,t)$ we obtain

$$d_h(x,s) \le -\sqrt{d_h^2(x,t) - 4(N-1)(s-t+h)} \tag{33}$$

for $t < s$ and $s + h - t < d_h^2(x,t)/(8(N+1))$.

4.3 Convergence of the scheme

Up to a subsequence we have

$$\overline{E}_{h_l} \xrightarrow{\mathcal{K}} E \qquad \text{and} \qquad (\mathring{E}_{h_l})^c \xrightarrow{\mathcal{K}} A^c$$

for a suitable closed sets E and a suitable open set $A \subset E$. Define $E(t)$ and $A(t)$ as in (29).

Observe that if $E(t) = \emptyset$ for some $t \ge 0$, then (32) implies that $E(s) = \emptyset$ for all $s \ge t$ so that we can define, as in Definition 2.1, the extinction time T^* of E, and similarly (in view of (33)) the extinction time ${T'}^*$ of A^c. Notice that at least one between T^* and ${T'}^*$ is $+\infty$. Possibly extracting a further subsequence, we have the following result:

Proposition 4.4. *There exists a countable set $\mathcal{N} \subset (0, +\infty)$ such that $d_{h_l}(\cdot, t)^+ \to \mathrm{dist}(\cdot, E(t))$ and $d_{h_l}(\cdot, t)^- \to \mathrm{dist}(\cdot, A^c(t))$ locally uniformly for all $t \in (0, +\infty) \setminus \mathcal{N}$.*

Moreover, for every $x \in \mathbb{R}^N$ the functions $\mathrm{dist}(x, E(\cdot))$ and $\mathrm{dist}(x, A^c(\cdot))$ are left continuous and right lower semicontinuous. Equivalently, the functions $E(\cdot)$ and $A^c(\cdot)$ are left continuous and right upper semicontinuous with respect to the Kuratowski convergence. Finally, $E(0) = E^0$ and $A(0) = \mathring{E}^0$.

Proof. By the Ascoli-Arzelà Theorem and a standard diagonal argument, we may extract a further (not relabeled) subsequence such that $d_{h_l}(\cdot, t) \to d(\cdot, t)$ locally uniformly for all $t \in \mathbb{Q} \cap (0, +\infty)$, where $d(\cdot, t)$ is either a Lipschitz function or infinite everywhere. In the latter case, either $d(\cdot, t) \equiv +\infty$ or $d(\cdot, t) \equiv -\infty$.

We observe that for all $t \in (0, T^*) \cap \mathbb{Q}$ we have $d(\cdot, t) < +\infty$. To see this we argue by contradiction assuming that for every $x \in \mathbb{R}^N$ and for every $M > 0$ we have $d_{h_l}(x,t) > M$ for all l large enough. We may now apply (32) to deduce that there exists a right interval (t, t') independent of l such that $d_{h_l}(x,s) > \frac{M}{2}$ for l large enough and for all $s \in (t,t')$; that is, $d_{h_l}(\cdot, s) \to +\infty$ for all $s \in (t,t')$. This in turn would imply $E(s) = \emptyset$ for all $s \in (t,t')$, which is impossible since $t < T^*$. Using (33) instead of (32) and arguing similarly, one can show that for all $t \in (0, {T'}^*) \cap \mathbb{Q}$ we have $d(\cdot, t) > -\infty$.

Assume that $d(x,t) > 0$, $t \in \mathbb{Q} \in (0,\infty)$. Then, $d_{h_l}(x,t) > 0$ for l large enough, so that by (32) $d_{h_l}(x,s) \geq \sqrt{d^2_{h_l}(x,t) - 4(N-1)(s-t+h_l)}$ for $t < s$ and $s + h_l - t < C(N)d^2_{h_l}(x,t)$, where we have set $C(N) := 1/(8(N+1))$. In turn, sending $l \to \infty$, we obtain

$$d(x,s) \geq \sqrt{d^2(x,t) - 4(N-1)(s-t)} \qquad \text{for } t,s \in \mathbb{Q} \text{ s.t. } 0 < s-t < C(N)d^2(x,t). \tag{34}$$

Symmetrically, using (33) in place of (32), we can deduce that if $d(x,t) < 0$, then

$$d(x,s) \leq -\sqrt{d^2(x,t) - 4(N-1)(s-t)} \qquad \text{for } t,s \in \mathbb{Q} \text{ s.t. } 0 < s-t < C(N)d^2(x,t). \tag{35}$$

Suppose now that $\limsup_{s\in\mathbb{Q},s\to t^+} d(x,s) =: R > 0$, and let $t_k \to t^+$ be a sequence of rational numbers such that $d(x,t_k) \to R$. Fix $s \in \mathbb{Q}$ such that $0 < s - t < \frac{C(N)}{4}R^2$. Since for k large enough we have $t < t_k < s$ and $d(x,t_k) \geq \frac{R}{2}$, we may apply (34) to deduce that for all such k's we have $d(x,s) \geq \sqrt{d^2(x,t_k) - 4(N-1)(s-t_k)}$. Sending $k \to \infty$ we obtain $d(x,s) \geq \sqrt{R^2 - 4(N-1)(s-t)}$ and, in turn,

$$\liminf_{s\in\mathbb{Q},s\to t^+} d(x,s) \geq \limsup_{s\in\mathbb{Q},s\to t^+} d(x,s)\,. \tag{36}$$

If $\liminf_{s\in\mathbb{Q},s\to t^+} d(x,s) < 0$, then we may argue in a similar way, using (35) instead of (34), to conclude that also in this case (36) holds. Summarizing, we have shown that $\lim_{s\in\mathbb{Q},s\to t^+} d(x,s)$ exists at all t. We denote by $d(x,t+0)$ such a right limit.

Assume now that $\limsup_{s\in\mathbb{Q},s\to t^-} d(x,s) =: R > 0$. Let $t_k \to t^-$ be a sequence of rational numbers such that $d(x,t_k) \to R$, $d(x,t_k) \geq \frac{R}{2}$ and $0 < t - t_k \leq \frac{C(N)}{4}R^2$ for all k. Analogously, let $s_j \to t^-$ be a sequence of rational numbers such that $d(x,s_j) \to \liminf_{s\in\mathbb{Q},s\to t^-} d(x,s)$. Fix $k \in \mathbb{N}$. Then, for all j large enough we have $t_k < s_j < t$, so that we may apply (34) and get $d(x,s_j) \geq \sqrt{d^2(x,t_k) - 4(N-1)(s_j - t_k)}$. Sending first $j \to \infty$ and then $k \to \infty$ we arrive at

$$\liminf_{s\in\mathbb{Q},s\to t^-} d(x,s) \geq \limsup_{s\in\mathbb{Q},s\to t^-} d(x,s)\,.$$

The same conclusion can be reached if $\liminf_{s\in\mathbb{Q},s\to t^-} d(x,s) < 0$, by arguing similarly and using (35) instead of (34). Summarizing, we have shown that also $\lim_{s\in Q,s\to t^-} d(x,s)$ exists at all t. We will denote by $d(x,t-0)$ such a left limit.

Suppose now that $\limsup_{l\to\infty,s\to t} d_{h_l}(x,s) =: R > 0$ and let $l_k \to \infty$ and $t_k \to t$ be such that $d_{h_{l_k}}(x,t_k) \to R$. Let $s \in \mathbb{Q}$ be such that $0 < s - t < \frac{C(N)}{8}R^2$. Then, for k sufficiently large we have that $t_k < s$, $|t_k - t| < \frac{C(N)}{8}R^2$,

$d_{h_{l_k}}(x,t_k) \geq \frac{R}{2}$, and $s + h_{l_k} - t_k \leq s + h_{l_k} - t + |t_k - t| < \frac{C(N)}{4}R^2$. We may then apply (32) to get $d_{h_{l_k}}(x,s) \geq \sqrt{d^2_{h_{l_k}}(x,t_k) - 4(N-1)(s + h_{l_k} - t_k)}$ for k sufficiently large. Sending $k \to \infty$ we deduce $d(x,s) \geq \sqrt{R^2 - 4(N-1)(s-t)}$. In turn, passing to the limit as $s \to t^+$, $s \in \mathbb{Q}$, we conclude that $d(x,t+0) \geq R = \limsup_{l\to\infty, s\to t} d_{h_l}(x,s)$. A similar argument shows that if $d(x,t-0) > 0$, then $\liminf_{l\to\infty,s\to t} d_{h_l}(x,s) \geq d(x,t-0)$. Arguing symmetrically (and using (33) instead of (32)), we can also show that if $\liminf_{l\to\infty,s\to t} d_{h_l}(x,s) < 0$, then $d(x,t+0) \leq \liminf_{l\to\infty,s\to t} d_{h_l}(x,s)$ and that if $d(x,t-0) < 0$, then $\limsup_{l\to\infty,s\to t} d_{h_l}(x,s) \leq d(x,t-0)$. All the above discussion can be summarized as follows:

$$\begin{aligned} d(x,t+0)^{\pm} &\geq \limsup_{l\to\infty,s\to t} d_{h_l}(x,s)^{\pm} \\ &\geq \liminf_{l\to\infty,s\to t} d_{h_l}(x,s)^{\pm} \geq d(x,t-0)^{\pm}. \end{aligned} \tag{37}$$

Let $\mathcal{N}$ be the set of all times t such that the left and right limits of d differ at (x,t), for some $x \in \mathbb{R}^N$ (we also assume $0 \in \mathcal{N}$). Notice that $\mathcal{N}$ is countable, since it can be written as the union over $k \in \mathbb{N}$ and $x \in \mathbb{Q}^N$ of the times such that the gap between the right and left limit of $d(x,\cdot)$ is larger than $1/k$ (which for k and x fixed cannot have cluster points). We denote by $d(x,t)$ the common value of the right and left limits of $d(x,\cdot)$ at $t \notin \mathcal{N}$.

By (37) we immediately have that $\lim_{l\to\infty} d_{h_l}(\cdot,t) = d(\cdot,t)$ for all $t \notin \mathcal{N}$. We now show that for $t \notin \mathcal{N}$, we have $d(\cdot,t)^+ = \mathrm{dist}(\cdot, E(t))$. This is equivalent to showing that $E(t)$ coincides with the Kuratowski limit K of $E_{h_l}(t)$, since $d(\cdot,t)^+ = \mathrm{dist}(\cdot,K)$. Clearly, $K \subseteq E(t)$. Conversely, if $x \notin K$, then $d(x,t)^+ =: R > 0$. Since d is continuous at t, we may find ε so small that $\lim_{l\to\infty} d_{h_l}(x,t-\varepsilon) \geq d(x,t-\varepsilon) > R/2$ and in turn, by (32), $W(x,R/4) \times [t-\varepsilon, t+\varepsilon] \cap E_{h_l} = \emptyset$ for l large enough. Thus $x \notin E(t)$, showing that $E(t) = K$ and $d(x,t)^+ = \mathrm{dist}(x,E(t))$. A similar argument (now relying on (33)) yields that $d(x,t)^- = \mathrm{dist}(x,A^c)$.

Always by (32), one can easily prove that $E(0) \subseteq E^0$. Since $E_{h_l}(0) = E^0$ for all l, we infer the equality $E(0) = E^0$. Symmetrically, using (33) one can show that $A(0) = \mathring{E}^0$.

Finally, we prove the continuity properties of $E(t)$ (the proof of the continuity properties of $A^c(t)$ being fully analogous). The right upper semicontinuity with respect to the Kuratowski convergence is a consequence of the fact that E is closed. Let us prove now the left continuity. To this aim, denote by $\hat{K}$ the Kuratowski limit of $E(s)$ as $s \nearrow t$. Clearly $\hat{K} \subseteq E(t)$. Let now $x \notin \hat{K}$. Then $\lim_{s\nearrow t} \mathrm{dist}(x,E(s)) = \mathrm{dist}(x,\hat{K}) =: R > 0$. Arguing exactly as before we may choose ε so small that $\liminf_l \mathrm{dist}(x, E_{h_l}(t-\varepsilon)) \geq \mathrm{dist}(x, E(t-\varepsilon)) > R/2$ and $W(x,R/4) \times [t-\varepsilon,t+\varepsilon] \cap E_{h_l} = \emptyset$ for all l large enough, so that $x \notin E(t)$. Hence $\hat{K} = E(t)$. This establishes the Kuratowski left-continuity of $E(\cdot)$ and concludes the proof of the proposition. □

Theorem 4.5. *The set E is a supersolution in the sense of Definition 2.1 with initial datum E^0, while A is a subsolution with initial datum E^0.*

Proof. Points *(a), (b)* and *(c)* of Definition 2.1 follow from Proposition 4.4. It remains to show *(d)*. Possibly extracting a further subsequence and setting $z_{h_l}(\cdot,t) := 0$ for $t > T^*_{h_l}$ if $T^*_{h_l} < T^*$, we may assume that z_{h_l} converges weakly-$*$ in $L^\infty(\mathbb{R}^N \times (0,T^*);\mathbb{R}^N)$ to some vector-field z satisfying $\phi^\circ(z) \le 1$ almost everywhere. Recall that by (28) we have $u_h^{k+1} \le d_{E_h^{k+1}}$, whenever $d_{E_h^{k+1}} \ge 0$. In turn, it follows from (25) that

$$\operatorname{div} z_h^{k+1} \le \frac{d_{E_h^{k+1}} - d_{E_h^k}}{h} \qquad \text{a.e. on } \{d_{E_h^{k+1}} \ge 0\}. \tag{38}$$

Consider a nonnegative test function $\eta \in C_c^\infty((\mathbb{R}^N \times (0,T^*)) \setminus E)$. If l is large enough, then the distance of the support of η from E_{h_l} is bounded away from zero. In particular, d_{h_l} is finite and positive on $\operatorname{Supp}\eta$. We deduce from (38) that

$$\begin{aligned}&\int\int \eta(x,t)\left(\frac{d_{h_l}(x,t+h_l) - d_{h_l}(x,t)}{h_l} - \operatorname{div} z_{h_l}(x,t+h_l)\right) dtdx\\ =& -\int\int \left(\frac{\eta(x,t)-\eta(x,t-h_l)}{h_l} d_{h_l}(x,t) - z_{h_l}(x,t+h_l)\cdot\nabla\eta(x,t)\right) dtdx \ge 0.\end{aligned}$$

Passing to the limit $l \to \infty$ we obtain (9).

Next, we establish an upper bound for $\operatorname{div} z_{h_l}$ away from E_{h_l}. To this aim observe that

$$d_{E_h^k} = \min_{y \in E_h^k} \phi^\circ(\cdot - y)$$

so that, by (25) and the comparison principle stated at the end of Proposition 4.1,

$$u_h^{k+1} \le \min_{y \in E_h^k} \phi_h^\circ(\cdot - y)$$

where ϕ_h° is given in (30). Thus, if $d_{E_h^k}(x) \ge R > 0$, then

$$u_h^{k+1}(x) \le \min_{y\in E_h^k} \phi^\circ(x-y) + h\frac{N-1}{R} = d_{E_h^k}(x) + h\frac{N-1}{R},$$

provided $h \le R^2/(N+1)$. As a consequence of (25), we obtain

$$\operatorname{div} z_h^{k+1} \le \frac{N-1}{R} \qquad \text{a.e. in } \{x :\ d_{E_h^k}(x) \ge R\}. \tag{39}$$

It is then easy to deduce from the convergence properties of E_{h_l} and d_{h_l} that

$$\operatorname{div} z \le \frac{N-1}{R} \qquad \text{in } \{(x,t) \in \mathbb{R}^N \times (0,T^*) :\ d(x,t) > R\}$$

in the sense of distributions. It follows that $\operatorname{div} z$ is a Radon measure in $\mathbb{R}^N \times (0,T^*) \setminus E$, and $(\operatorname{div} z)^+ \in L^\infty(\{(x,t) \in \mathbb{R}^N \times (0,T^*) :\ d(x,t) \ge \delta\})$ for every $\delta > 0$.

We now provide a lower (h-dependent) bound for $\operatorname{div} z_{h_l}$. To this aim, note that if $d_{E_h^k}(x) =: R > 0$, then $d_{E_h^k} \geq R - \phi^\circ(\cdot - x)$. Thus, by comparison as before,

$$u_h^{k+1}(x) \geq R - \phi_h^\circ(0) = R - \sqrt{h}\frac{2N}{\sqrt{N+1}}\,.$$

In turn, by (25), we deduce

$$\operatorname{div} z_h^{k+1} \geq -\frac{1}{\sqrt{h}}\frac{2N}{\sqrt{N+1}} \qquad \text{a.e. in } \{x : d_{E_h^k}(x) > 0\}.$$

Combining the above inequality with (39) and using (25) again, we deduce that for all $t \in (0, T^*) \setminus \mathcal{N}$ (where recall that $\mathcal{N}$ is introduced in Proposition 4.4) and any $\delta > 0$

$$\|u_{h_l}(\cdot, t) - d_{h_l}(\cdot, t - h_l)\|_{L^\infty(\{x: d_{h_l}(x, t-h_l) \geq \delta\})} \leq \sqrt{h_l}\frac{2N}{\sqrt{N+1}},$$

provided that l is large enough. In particular, recalling the convergence properties of E_{h_l} and d_{h_l} (see also (37)), we deduce that

$$u_{h_l} \to d \qquad \text{a.e. in } \mathbb{R}^N \times (0, T^*) \setminus E, \tag{40}$$

with the sequence $\{u_{h_l}\}$ locally (in space and time) uniformly bounded.

Consider now, as before, a nonnegative test function $\eta \in C_c^\infty((\mathbb{R}^N \times (0, T^*)) \setminus E)$. Then, recalling (40), we have by lower semicontinuity

$$\int\!\!\int \phi(\nabla d)\eta \, dxdt \leq \liminf_l \int\!\!\int \phi(\nabla u_{h_l})\eta \, dxdt = \liminf_l \int\!\!\int (z_{h_l} \cdot \nabla u_{h_l})\eta \, dxdt.$$

On the other hand,

$$\int\!\!\int (z_{h_l} \cdot \nabla u_{h_l})\eta \, dxdt = \int\!\!\int (z_{h_l} \cdot \nabla d)\eta \, dxdt + \int\!\!\int z_{h_l} \cdot \nabla(u_{h_l} - d)\eta \, dxdt,$$

with

$$\int\!\!\int (z_{h_l} \cdot \nabla d)\eta \, dxdt \overset{l\to\infty}{\longrightarrow} \int\!\!\int (z \cdot \nabla d)\eta \, dxdt.$$

Hence, we obtain

$$\int\!\!\int \phi(\nabla d)\eta \, dxdt \leq \int\!\!\int (z \cdot \nabla d)\eta \, dxdt, \tag{41}$$

provided we show that

$$\lim_l \int\!\!\int z_{h_l} \cdot \nabla(u_{h_l} - d)\eta \, dxdt = 0. \tag{42}$$

For each t, set

$$m_l(t) := \min_{x \in \operatorname{Supp} \eta(\cdot, t)} \big(u_{h_l}(x, t) - d(x, t)\big), \quad M_l(t) := \max_{x \in \operatorname{Supp} \eta(\cdot, t)} \big(u_{h_l}(x, t) - d(x, t)\big).$$

Recall that these quantities are uniformly bounded and converge to 0 at all $t \notin \mathcal{N}$. Then, we can write

$$\int\int z_{h_l} \cdot \nabla(u_{h_l} - d)\eta \, dxdt = \int\int z_{h_l} \cdot \nabla(u_{h_l} - d - m_l)\eta \, dxdt$$
$$= -\int\int (u_{h_l} - d - m_l)(z_{h_l} \cdot \nabla\eta + \eta \mathrm{div}\, z_{h_l}) \, dxdt. \quad (43)$$

For l large enough, since the support of η is at positive distance from E there exists $\delta > 0$ such that $d_{h_l} \geq \delta$ everywhere on this support, so that $\mathrm{div}\, z_{h_l} \leq (N-1)/\delta$. It follows that

$$-\int\int (u_{h_l} - d - m_l)\eta \mathrm{div}\, z_{h_l} \, dxdt \geq -\frac{N-1}{\delta}\int\int (u_{h_l} - d - m_l)\eta \, dxdt \stackrel{l\to\infty}{\longrightarrow} 0,$$

thanks also to (40). Recalling (43), we can conclude that

$$\liminf_l \int\int z_{h_l} \cdot \nabla(u_{h_l} - d)\eta \, dxdt \geq 0.$$

In the same way, writing now

$$\int\int z_{h_l} \cdot \nabla(u_{h_l} - d)\eta \, dxdt = \int\int z_{h_l} \cdot \nabla(u_{h_l} - d - M_l)\eta \, dxdt$$

and using $u_{h_l} - d - M_l \leq 0$ a.e. on $\mathrm{Supp}\,\eta$, one can show that

$$\limsup_l \int\int z_{h_l} \cdot \nabla(u_{h_l} - d)\eta \, dxdt \leq 0$$

so that (42) follows. In turn, (41) holds, that is, $\phi(\nabla d) \leq z \cdot \nabla d$ a.e. in $\mathbb{R}^N \times (0, T^*) \setminus E$. On the other hand, recalling that $\phi^\circ(z) \leq 1$ a.e. in $\mathbb{R}^N \times (0, T^*)$, we have

$$z \cdot \nabla d \leq \phi(\nabla d)$$

a.e. in $\mathbb{R}^N \times (0, T^*)$. We conclude that $\phi(\nabla d) = z \cdot \nabla d$ and, in turn, $z \in \partial\phi(\nabla d)$ a.e. in $\mathbb{R}^N \times (0, T^*) \setminus E$. This concludes the proof that E is a supersolution. The proof that A is a subsolution is identical. $\square$

Corollary 4.6. *Let u^0 be a bounded, uniformly continuous in $\mathbb{R}^N$. Then for all $s \in \mathbb{R}$ but a countable number, the minimizing movement scheme starting from $E^0_s = \{u^0 \leq s\}$ converges to the unique solution of the curvature flow in the sense of Definition 2.1, with initial datum E^0_s.*

Proof. The arguments are standard and rely on the comparison theorem 3.3. The essential point is that since u^0 is uniformly continuous, then for $s \neq s'$ the sets ∂E^0_s and $\partial E^0_{s'}$ are at positive distance, so that thanks to Theorems 4.5 and 3.3, limits of discrete flows starting from each of these two sets also remain at (the same) positive distance. The bad set is the set of levels for which “fattening” occurs, that is, $|E \setminus A| > 0$. The embedding of the level sets (and the fact that the fattening must last for a positive time) yields that this may happen only for a countable number of levels. For any other level there is only one possible limit, showing that the minimizing movement scheme is converging. This construction also provides a unique level-set solution $u(x,t)$ starting from u^0, which shares the same spatial modulus of continuity and is also uniformly continuous in time. (*cf* for instance [16, 19]). □

Remark 4.7 (Star-shaped initial sets and graphs). A natural issue is to understand under which circumstances fattening does not occur. To our knowledge, there is no general result, even for the classical mean curvature flow. On the other hand, it is classical [47, Sec. 9] that strictly star-shaped sets do not develop fattening and the proof in [47] is valid in our setting. In the same way,if ∂E_0 is the graph of a uniformly continuous functions, similar arguments will show that it cannot develop fattening and that the evolution from E_0 is unique and remains a graph (with the same modulus of continuity) for all time, as in the classical case [23, 25].

5 Conclusion and perspectives

In this paper we have shown the existence and uniqueness of a mean curvature flow (namely, the "natural" flow by mean curvature along the Cahn-Hoffmann vector field) with a technique which does not require any type of regularity on the surface tension, and thus have provided the first sound definition of a crystalline curvature flow in any dimension. It does not require that the initial surface is bounded and applies, in particular, also to the case of graphs. The uniqueness result is based on a very standard parabolic comparison principle. The general approach, based on the fact that the level sets of the distance functions have nonincreasing curvatures as the distance increases (as was exploited as early as in [27, 47] in the viscosity setting), can quite probably be used in more general situations, and even maybe for motions which are not necessarily variational. However, it should need substantial adaption. For instance, if replacing the mobility $m = \phi^\circ$ in our approach by other (convex) functions is in principle easy (it is enough to consider, for the distance functions, the m-distance function instead of the ϕ°-distance), in the nonsmooth case it yields difficulties which still require further investigation. Indeed, if m is smooth and ϕ is not, then it will not be true anymore that the level sets of the distance function have globally bounded curvature as the distance increases, so that Definition 2.1 needs to be changed. It is not yet clear what assumption on $(\operatorname{div} z)^\pm$ is then useful in order to be able to derive both existence and uniqueness. We will address this issue in a forthcoming paper.

A Superflows and viscosity supersolutions

In this Appendix, we prove briefly the assertion in Remark 2.3: when $\phi \in C^2(\mathbb{R}^N \setminus \{0\})$, we claim that a superflow/supersolution in our sense is also a viscosity supersolution of the corresponding geometric partial differential equation:

$$\partial_t d = \phi(\nabla d)\mathrm{div}\,\nabla\phi(\nabla d). \tag{44}$$

In particular, in that case, comparison and existence of solutions follows from the standard theory [21]. In fact, the following holds:

Proposition A.1. *Assume $\phi \in C^2(\mathbb{R}^N \setminus \{0\})$. Then, if E is a superflow in the sense of Definition 2.1, with extinction time $T^* \in (0, +\infty]$, the function $d(x,t) := \mathrm{dist}(x, E(t))$ is a lower semicontinuous viscosity supersolution of* (44) *in $\mathbb{R}^N \times (0, T^*)$. If in addition, $\phi^\circ \in C^2(\mathbb{R}^N \setminus \{0\})$, then the converse holds.*

Proof. Consider a supersolution in the sense of Definition 2.1. A first remark is that (see Lemma 2.4) d is lower semicontinuous. Let ψ be a smooth test function and assume $d - \psi$ has a strict local minimum at a point $(\bar{x}, \bar{t})$, with $d(\bar{x}, \bar{t}) = \psi(\bar{x}, \bar{t}) > 0$. First of all, it is standard that in this case one must be at a point of differentiability for d, where $\phi(\nabla\psi(\bar{x}, \bar{t})) = \phi(\nabla d(\bar{x}, \bar{t})) = 1$. Let us assume, by contradiction, that in $(\bar{x}, \bar{t})$,

$$\partial_t \psi < \phi(\nabla\psi)\mathrm{div}\,\nabla\phi(\nabla\psi) = \mathrm{div}\,\nabla\phi(\nabla\psi),$$

so that by continuity of ψ and $\nabla\phi$, $D^2\phi$ we may assume that the strict inequality also holds in a neighborhood $B = B_\delta(\bar{x}, \bar{t})$ of $(\bar{x}, \bar{t})$. Assume in addition δ and $\varepsilon > 0$ are such that $d(x,t) - \psi(x,t) \geq 2\varepsilon$ in $B_\delta(\bar{x}, \bar{t}) \setminus B_{\delta/2}(\bar{x}, \bar{t})$ and consider, given $\Psi \in C^\infty(\mathbb{R})$, nonincreasing, convex, vanishing on $\mathbb{R}_+$ and positive on $(-\infty, 0)$, the function $w = \Psi(d - \psi - \varepsilon)\chi_B$. It is BV in time, Lipschitz in space, and compactly supported in $B_{\delta/2}(\bar{x}, \bar{t})$, moreover $w(\bar{x}, \bar{t}) = \Psi(-\varepsilon) > 0$ (and this also holds in a neighborhood of $\bar{x}$, by continuity). As in the proofs of Lemma 3.2 and Theorem 3.3, we can compute (using $\Psi' \leq 0$ and the chain rule for BV functions)

$$\partial_t w \leq \Psi'(d - \psi - \varepsilon)(\partial_t^d d - \partial_t \psi) \leq \Psi'(d - \psi - \varepsilon)(\mathrm{div}\,\nabla\phi(\nabla d) - \mathrm{div}\,\nabla\phi(\nabla\psi))$$

where the inequality holds as measures in B. Hence,

$$\begin{aligned}\partial_t \int w\,dx &\leq \int \chi_B \Psi'(d - \psi - \varepsilon)\mathrm{div}\,(\nabla\phi(\nabla d) - \nabla\phi(\nabla\psi)) \\ &= -\int \chi_B \Psi''(d - \psi - \varepsilon)(\nabla\phi(\nabla d) - \nabla\phi(\nabla\psi) \cdot (\nabla d - \nabla\psi)\,dx \leq 0.\end{aligned}$$

However, for $t = \bar{t} - \delta/2$, $w(\cdot, t) \equiv 0$, hence one cannot have $w(\cdot, \bar{t}) > 0$ near $\bar{x}$: a contradiction. We deduce

$$\partial_t \psi(\bar{x}, \bar{t}) \geq \phi(\nabla\psi(\bar{x}, \bar{t}))\mathrm{div}\,\nabla\phi(\nabla\psi)(\bar{x}, \bar{t}) \tag{45}$$

so that we have shown that d is a viscosity supersolution in $\{d > 0\}$.

If now, $d - \psi$ has a strict local minimum at $(\bar{x}, \bar{t})$ with $d(\bar{x}, \bar{t}) = \psi(\bar{x}, \bar{t}) = 0$, there are two situations. In case $\nabla\psi(\bar{x}, \bar{t}) = 0$, it is standard that one may also assume that $D^2\psi$ vanishes at this point (see [40]). Then, one can prove that the estimate in Lemma 3.2 will lead to a contradiction if $\partial_t\psi(\bar{x},\bar{t}) < 0$. Hence $\partial_t\psi(\bar{x},\bar{t}) \geq 0$.

In case $\bar{p} = \nabla\psi(\bar{x},\bar{t}) \neq 0$, then near this point, $\{\psi \leq 0\}$ is a smooth set which contains E, with a contact at $(\bar{x},\bar{t})$. Then, $\delta(x,t) = \mathrm{dist}(x, \{\psi \leq 0\})$ is a new, smooth function near $(\bar{x},\bar{t})$ such that $d - \delta$ has a local minimum, not only at $(\bar{x},\bar{t})$, but also at $(\bar{x}_s,\bar{t})$ for $\bar{x}_s = \bar{x} + s\nabla\phi(\bar{p})$, $s > 0$ small. It follows (as in (45)) that $\partial_t\delta(\bar{x}_s,\bar{t}) \geq \mathrm{div}\,\nabla\phi(\nabla\delta)(\bar{x}_s,\bar{t})$ and letting then $s \to 0$, we deduce that (45) holds again. Hence, d is a viscosity supersolution of the geometric equation (44).

Conversely, assuming now that ϕ° is also C^2, consider $E \subset \mathbb{R}^N \times [0,T)$ such that $d(x,t) := d_{E(t)}(x)$ is a (lower semicontinuous) viscosity supersolution of (45).

A first step is to show, by standard comparison (with explicit solutions) that an inequality such as (33) still holds: for any t, τ, x:

$$d(x,t+\tau)^2 \geq d(x,\tau)^2 - c\tau$$

for some constant c (in other words the function $d(\cdot,t)^2 + ct$ is nondecreasing). We omit the proof of this point. One can deduce that $\partial_t d \geq -c/\delta$ in $\{d > \delta\}$, for some constant c, in the distributional sense. Property (c) in Definition 2.1 easily follows, using the fact that d is lower semicontinuous. We need now to show (d).

As the property is local, we can work in a small ball $B \subset\subset \{d > \delta\}$. A first observation is that the regularity of ϕ° implies that d is semiconcave in the x variable in B, thanks to the inequality:

$$\begin{aligned} d(x+h,t) - 2d(x,t) + d(x-h,t) &\leq \phi^\circ(x+h-y) - 2\phi^\circ(x-y) + \phi^\circ(x-h-y) \\ &= \int_{-1}^{1} (1-|s|)(D^2\phi^\circ(x-y+sh)h)\cdot h\,ds \\ &\leq c|h|^2 \int_{-1}^{1} \frac{1-|s|}{\phi^\circ(x-y+sh)}\,ds \leq \frac{c}{\delta}|h|^2, \end{aligned}$$

where y is the projection of x on $\partial E(t)$ provided h is small enough ($\phi^\circ(h) \leq \delta/2$). We have used here the fact that $\phi^\circ D^2\phi^\circ$ is zero-homogeneous and bounded.

For $\varepsilon > 0$, we introduce

$$d^\varepsilon(x,t) = \min_{s\leq t} d(x,t-s) + \frac{s^2}{2\varepsilon}. \tag{46}$$

We assume ε is small enough, so that the minimum is always reached at a point $(x,t-s) \in \{d > \delta\}$, for $(x,t) \in B$. Then it is standard that d^ε is semiconcave in B: if $(x \pm h, t \pm \tau) \in B$ and s reaches the minimum in (46), we have now

$$\begin{aligned} &d^\varepsilon(x+h,t+\tau) - 2d^\varepsilon(x,t) + d^\varepsilon(x-h,t-\tau) \\ &\leq d(x+h,t-s) + \frac{(s+\tau)^2}{2\varepsilon} - 2d(x,t-s) - \frac{s^2}{\varepsilon} + d(x-h,t-s) + \frac{(s-\tau)^2}{2\varepsilon} \\ &\leq \frac{c}{\delta}h^2 + \frac{1}{\varepsilon}\tau^2, \end{aligned}$$

again if $\phi^\circ(h)$ is small enough. As an infimum of supersolutions, it is also a viscosity supersolution (at least if ε is small enough, so that the minimum in (46) is reached for $t-s>0$). Using Aleksandrov's theorem (see [22] and the versions in [46] and [43]), one has at a.e. $(x,t)\in B$ a second order jet:

$$\begin{aligned} d^\varepsilon(x+h,t+\tau) &= d^\varepsilon(x,t)+\partial_t d^\varepsilon(x,t)\tau+\nabla d^\varepsilon(x,t)\cdot h+\frac{1}{2}(D^2 d^\varepsilon(x,t)h)\cdot h \\ &\quad +\frac{1}{2}\partial^2_{tt}d^\varepsilon(x,t)\tau^2+\tau\partial_t\nabla d^\varepsilon(x,t)\cdot h+o(\tau^2+|h|^2) \qquad (47)\\ &= d^\varepsilon(x,t)+\partial_t d^\varepsilon(x,t)\tau+\nabla d^\varepsilon(x,t)\cdot h+\frac{1}{2}(\nabla^2 d^\varepsilon(x,t)h)\cdot h \\ &\quad + o(|\tau|+|h|^2). \end{aligned}$$

(Clearly, the corresponding functions $\nabla d^\varepsilon(x,t)$, $\nabla^2 d^\varepsilon(x,t)$, $\partial_t d^\varepsilon(x,t)$ and $\partial_t\nabla d^\varepsilon(x,t)$, defined a.e. in B, must be measurable.)

Being d^ε a viscosity supersolution, it follows at such a point that

$$\partial_t d^\varepsilon(x,t)\ge\phi(\nabla d^\varepsilon)D^2\phi(\nabla d^\varepsilon(x,t)):\nabla^2 d^\varepsilon(x,t). \qquad (48)$$

On the other hand, letting $z^\varepsilon(x,t):=\nabla\phi(\nabla d^\varepsilon(x,t))$, if ρ_η is a (spatio-temporal) smoothing kernel and $d^\varepsilon_\eta=\rho_\eta * d^\varepsilon$, $z^\varepsilon_\eta=\nabla\phi(\nabla d^\varepsilon_\eta)$, we have

$$\operatorname{div} z^\varepsilon_\eta = D^2\phi(\nabla d^\varepsilon_\eta):D^2 d^\varepsilon_\eta.$$

Since $z^\varepsilon_\eta\to z^\varepsilon$ as $\eta\to 0$ (for instance in any L^p, $p<\infty$), then $\operatorname{div} z^\varepsilon_\eta\to\operatorname{div} z^\varepsilon$ in $\mathcal{D}'(B)$. On the other hand, as $D^2 d^\varepsilon$ is a Radon measure which is bounded from above, its singular part is nonpositive and it follows from (47) and Radon-Nykodym's theorem that

$$D^2 d^\varepsilon = \nabla^2 d^\varepsilon(x,t)dxdt+(D^2 d^\varepsilon)^s\le\nabla^2 d^\varepsilon(x,t)dxdt.$$

Hence,

$$D^2\phi(\nabla d^\varepsilon_\eta):D^2 d^\varepsilon_\eta\le D^2\phi(\nabla d^\varepsilon_\eta):(\rho_\eta * \nabla^2 d^\varepsilon)$$

and in the limit we obtain that

$$\operatorname{div} z^\varepsilon\le D^2\phi(\nabla d^\varepsilon(x,t)):\nabla^2 d^\varepsilon(x,t)dxdt$$

as measures in B. Then, (48) implies

$$\partial_t d^\varepsilon(x,t)\ge\phi(\nabla d^\varepsilon)\operatorname{div} z^\varepsilon. \qquad (49)$$

Observe that, given $(x,t)\in B$, if s reaches the minimum in (46) and if $p\in\partial^+ d(x,t-s)$ (the supergradient at x of $d(\cdot,t-s)$, which is semiconcave), then for any h small,

$$\begin{aligned} d^\varepsilon(x+h,t) &\le d(x+h,t-s)+\frac{s^2}{2\varepsilon} \\ &\le d(x,t-s)+p\cdot h+\frac{c}{\delta}|h|^2+\frac{s^2}{2\varepsilon}=d^\varepsilon(x,t)+p\cdot h+\frac{c}{\delta}|h|^2 \end{aligned}$$

so that $p \in \partial^+ d^\varepsilon(x,t)$. If in addition $d^\varepsilon(\cdot,t)$ is differentiable at x, it follows that $p = \nabla d^\varepsilon(x,t)$ must be the unique point in $\partial^+ d(x,t-s)$, and therefore equal to $\nabla d(x,t-s)$. As a consequence, $\phi(\nabla d^\varepsilon(x,t)) = 1$ a.e. in B.

Sending ε to zero in (49), we deduce that, still as measures in B,

$$\partial_t d \geq \operatorname{div} \tilde{z}$$

where $\tilde{z}$ is a L^∞ weak-$*$ limit of z^ε as $\varepsilon \to 0$. It remains to check that $\tilde{z} = z$: in fact, this easily follows from simple convexity arguments. At a continuity point t of d, one has that $d^\varepsilon(\cdot,t) \to d(\cdot,t)$, as $\varepsilon \to 0$, uniformly in space (in B), moreover these functions are semiconcave all with the same constant. It follows that $\nabla d^\varepsilon(\cdot,t) \to \nabla d(\cdot,t)$ a.e. as $\varepsilon \to 0$. Lebesgue's theorem ensures then that $\nabla d^\varepsilon \to \nabla d$ in $L^p(B)$ for all $p < \infty$. By continuity of $\nabla\phi$, it follows $z^\varepsilon \to z$. Hence $\tilde{z} = z$. □

It is unclear whether this is still true without the assumption that ϕ° is C^2. However, as we still expect that $\operatorname{div} z \leq (N-1)/d$ in this case, it could hold as well.

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